Session Six Overview

Making Meaning for Operations: Session 6

Agenda

Sharing Exit Card Comments	Whole group	5 minutes
Sharing Student Thinking Assignments	Small groups	25 minutes
Chapter 6 Case Discussion	Small groups	30 minutes
	Whole group	25 minutes
Break		15 minutes
Math Activity: Fractions in Division	Small groups	25 minutes
	Whole group	25 minutes
	Small groups	25 minutes
Homework and Exit Cards	Whole group	5 minutes

Mathematical Themes

- It may be necessary to expand ideas about multiplication of whole numbers in order to develop meaning for multiplication involving numbers less than 1.
- Just as multiplication of whole numbers can be represented as a rectangle, so can multiplication involving fractions and mixed numbers.
- Mapping a diagram solution for a division of fraction problem to the arithmetic procedures for the calculation provides access to understanding why $a \times b/c$ produces the same answer as $a \div c/b$.

Connections to the Common Core: Standards for Mathematical Practice

MP2 Reason abstractly and quantitatively.

MP3 Construct viable arguments and critique the reasoning of others.

MP6 Attend to precision.

MP7 Look for and make use of structure.

Connections to the Common Core: Content Standards

Grade 4: Number and Operations - Fractions 4
Grade 5: Number and Operations - Fractions 4, 7
Grade 6: The Number System - 1

4. NF. 4. *Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.*

a. Understand a fraction a/b as a multiple of 1/b. For example, use a visual fraction model to represent 5/4 as the product 5 x (1/4), recording the conclusion by the equation 5/4 = 5 x (1/4).

b. Understand a multiple of a/b as a multiple of 1/b, and use this understanding to multiply a fraction by a whole number. For example, use a visual fraction model to express 3 x (2/5) as 6 x (1/5), recognizing this product as 6/5. (In general, n x (a/b) = (n x a)/b.)

c. Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem. For example, if each person at a party will eat 3/8 of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?

5. NF. 4. *Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.*

a. Interpret the product (a/b) x q as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations a x q ÷ b. For example, use a visual fraction model to show (2/3) x 4 = 8/3, and create a story context for this equation. Do the same with (2/3) x (4/5) = 8/15. (In general, (a/b) x (c/d) = ac/bd.)

b. Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

5. NF. 7. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.1 a. Interpret division of a unit fraction by a non-zero whole number, and compute such quotients. For example, create a story context for (1/3) ÷ 4, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that (1/3) ÷ 4 = 1/12 because (1/12) x 4 = 1/3. b. Interpret division of a whole number by a unit fraction, and compute such quotients. For example, create a story context for 4 ÷ (1/5), and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that 4÷ (1/5) = 20 because 20 x (1/5) = 4. c. Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem. For example, how much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 1/3-cup servings are in 2 cups of raisins?

6. NS.1. *Apply and extend previous understandings of multiplication and division to divide fractions by fractions. 1. Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for (2/3) ÷ (3/4) and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that (2/3) ÷ (3/4) = 8/9 because 3/4 of 8/9 is 2/3. (In general, (a/b) ÷ (c/d) = ad/bc.) How much chocolate will*

each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?

Major Focus: *Apply and extend understanding of whole numbers to fractions.*

In Session Six, participants continue to engage with an overarching idea in the Common Core: As the number system is extended from whole numbers to fractions, ideas that were consolidated for whole numbers are revisited to see what stays the same and what needs to be modified to incorporate these new kinds of numbers. In this session, participants will examine multiplication and division with fractions.

Notes to the Facilitator regarding the Standards for Mathematical Practice

MP2 Reason abstractly and quantitatively. *Mathematically proficient students at the elementary grades make sense of quantities and their relationships in problem situations. They can contextualize quantities and operations by using images or stories. They interpret symbols as having meaning, not just as directions to carry out a procedure. Even as they manipulate the symbols, they can pause as needed to access the meaning of the numbers, the units, and the operations that the symbols represent. Mathematically proficient students know and flexibly use different properties of operations, numbers, and geometric objects.*

In the Math Activity, *Fractions in Division*, participants are asked to make connections among the quantities they have represented in the diagram, the context, and the arithmetic expressions they have composed. Reasoning with the context in mind should support participants in understanding why 6 2/3 is the correct answer to the division problem and what the two numbers, 6 and 2/5, represent in the situation. After discussing the mathematics, ask participants to reflect on MP2.

MP3 Construct viable arguments and critique the reasoning of others. *Mathematically proficient students at the elementary grades construct mathematical arguments—that is, explain the reasoning underlying a strategy, solution, or conjecture—using concrete referents such as objects, drawings, diagrams, and actions…. Mathematically proficient students can listen to or read the arguments of others, decide whether they make sense, ask useful questions to clarify or improve the arguments, and build on those arguments. They can communicate their arguments, compare them to others, and reconsider their own arguments in response to the critiques of others.*

The math activity leads participants toward constructing arguments for why the division algorithm works. Comparing their reasoning with the arguments of their colleagues is an example of MP3. The passage in Maxine's Journal, *Math activity: fractions in division*, offers an example of this discussion. Maxine does not cite MP3; however, you can make that connection explicit for your participants.

MP6 Attend to precision. *Mathematically proficient students at the elementary grades communicate precisely to others. They start by using everyday language to express their mathematical ideas, realizing that they need to select words with clarity and specificity rather than saying, for example, "it works" without explaining what "it" means. As they encounter the ambiguity of everyday terms, they come to appreciate, understand, and use mathematical vocabulary. Once young students become familiar with a mathematical idea or object, they are ready to learn more precise mathematical terms to describe it. In using representations, such as pictures, tables, graphs, or diagrams, they use appropriate labels to communicate the meaning of their representation.*

Articulating arguments requires carefully attention to mathematical language and making connections between the components of the argument and their representations. In the math activity, encourage participants to ask each other questions when explanations about why the answer is 6 2/5 and not 6 2/3 are confusing or unclear. Suggest they map elements of their arithmetic to the diagram and story as ways to clarify. Highlight such attempts at clarifying as examples of MP6.

MP7 Look for and make use of structure. M*athematically proficient students at the elementary grades use structures such as place value, the properties of operations, other generalizations about the behavior of the operations (for example, the less you subtract, the greater the difference), and attributes of shapes to solve problems.*

The mathematical structures that make up a rectangle are sometimes hidden from view. The students in case 26, *Multiplication of mixed numbers* make this structure explicit and use it to reason about product of 2 ¾ and 3 2/3. Basimah draws a 4 by 3 rectangle partitioned into 12 equal squares to show what part of the 3 x 4 rectangle is covered by 2 ¾ x 3 2/3. Partitioning the squares in the right hand column each into 3 equal rectangles and partitioning the four squares into the bottom row each into 4 equal rectangles, she is able to use these structures to label the fractional parts of the computation in reference to one of the 12 squares. Cases 23 and 34 contain additional examples of students making use of structure to reason about fractions.

NCTM's Mathematics Teaching Practices

While the focus on eliciting and following student thinking continues through all sessions of *Making Meaning for Operations*, discussion of teacher moves and teaching practice is also a component of a DMI seminar. Before the end of this session, take a few minutes to distribute the list of Mathematics Teaching Practices copied below. This list is from the National Council of Teachers of Mathematics 2014 publication *Principles to Actions*. Participants will use this document as part of their homework for Session Seven.

MMO Session Six Agenda Changes linked to Common Core

There are seven modifications to the agenda in Session Six.

1. Use a few minutes at the beginning to the session to share exit card comments.

2. Distribute the session overview.

3. Before the small groups start to share their student thinking assignment, remind the group to include discussion of the mathematical practice(s) they noted in their students' thinking.

4. The Focus Questions for the case discussion have been revised to include references to a document from the Common Core, *TABLE 2, Common multiplication and division situations*. See below for a copy of the revised Focus Questions and a copy of this table to distribute.

5. During the small and whole group discussion of the Focus Questions, provide an opportunity for participants to share the problems they generated as examples of multiplicative comparison and collect comments regarding the value of bringing all of these kinds of situations to their students.

6. *Math Activity: Fractions in Division* has been revised to include reference to *Table 2, Common Multiplication and Division Situations*. See below for a copy of the revised Math Activity.

7. The Seventh Homework has been revised in two ways. The reading assignment includes reference to NCTM's Mathematics Teaching Practices. The writing assignment has been revised to build specifically on the math activity from this session. See below for a copy of NCTM's Mathematics Teaching Practices and the revised Seventh Homework.

SESSION 6

Focus Questions: Chapter 6

Reflection: Examine *TABLE 2, Common multiplication and division situations*. Refer to this table as you discuss questions #1 and 2.

1. In his case 27, Henry states that "multiplication is not just repeated addition." What does he mean? What else is it?

2. Henry also indicates that it is difficult for his students to make sense of $1/2 \times 1/4$ or 0.6×0.5. We have been exploring the use of contexts and story situations to interpret arithmetic expressions. What problem situations would match these arithmetic expressions? Can you make more than one problem for each expression?

Additional Reflection on TABLE 2: The cases in *Making Meaning for Operations* include examples of the situations referred to in the table as "Equal Groups" and "Arrays, Area." Generate examples of your own to illustrate "Comparison." What opportunities can you create for your students to be sure they encounter all of these kinds of situations?

3. In Ann's case 25, Ferris, Liam, and Midori all use expressions containing what Ann calls "a portion of a portion equivalency." Explain the thinking of each student. What is correct about it? How does it connect with multiplication of fractions?

4. In the seminar, you have been drawing diagrams to represent multiplication involving fractions between 0 and 1. Sarita's students in case 26 also approach multiplication of mixed numbers with a diagram solution. What does Basimah figure out and show Sarita in the diagram after line 105? Try the same approach with some other pairs of numbers. Will it always work?

SESSION 6

Math Activity: Fractions in Division

1. Wanda really likes cake. She decides that one serving should be 3/5 of a cake. She has 4 cakes, all the same size. How many servings does she have?

    (a) Draw a diagram to model this situation.

    (b) What answer does your diagram indicate?

    (c) Solve the problem using an arithmetic sentence.

    (d) How does your arithmetic sentence match your diagram?

    (e) After 6 servings are eaten, how much cake is left?

    (f) For the division problem 4 ÷ 3/5, why is the answer 6 2/3 rather than 6 2/5?

Solve each of the following problems with a diagram. Write an arithmetic sentence that matches the situation. What connections do you see between the diagram and the arithmetic?

2. You are giving a birthday party. From Ben and Jerry's ice cream factory, you order 6 pints of ice cream. If you serve 3/4 pint of ice cream to each person, how many can you serve?
3. I am making a soup that requires 5/6 cups of green beans per person. If I have picked 3 cups of green beans from my garden, how many people will I be able to serve?

Solve each of the following problems with a diagram. Write an arithmetic sentence that matches the situation. What connections do you see between the diagram and the arithmetic? How are these two problems the same and how are they different? How would you categorize each problem using Table 2?

4. I eat 2/3 cup of cottage cheese for lunch each day. I have 2 2/3 cups of cottage cheese in my refrigerator. How long will it last me?
5. I put 2 2/3 gallons of gas into my empty lawn mower. I notice it is now 2/3 full. What is the capacity of the gas tank?

Seventh Homework

Reading assignment: Casebook chapter 7

In the casebook, read chapter 7, "Expanding Ideas About Division in the Context of Fractions," including the introductory text and cases 28 and 29. Use the questions posed in the introduction of the casebook to guide your reading.

Considering Teacher Moves: Selena, the teacher in Case 29, makes strategic moves to draw her students' attention to particular mathematical issues. Some of these decisions are seen in the questions she poses to the class; others are seen in classroom environment she has set up. As you read the case, take note of specific decisions or actions of the teacher. Use the list of NCTM's Mathematical Teaching Practices to categorize the teacher moves. If you have identified teacher moves that do not appear to be on this list, characterize them with your own categories. Mark each move with a sticky note or make use of relevant line numbers so you can easily locate them at the next session.

Writing assignment: Pursuing a mathematical question

This assignment is about the math you are learning in the seminar, not about the math learning of your students. The session included work on the following pair of problems:

> I eat 2/3 cup of cottage cheese for lunch each day. I have 2 2/3 cups of cottage cheese in my refrigerator. How long will it last me?
>
> I put 2 2/3 gallons of gas into my empty lawn mower. I notice it is now 2/3 full. What is the capacity of the gas tank?

Solve each with a diagram, create an arithmetic sentence that matches the situation, and create an arithmetic sentence that matches your diagram. What connections do you see between the diagram and the arithmetic?

Then respond to the following questions: How are these two problems the same and how are they different? How would you categorize each problem using Table 2?

NCTM's Mathematics Teaching Practices [1]

Establish mathematics goals to focus learning.

Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.

Implement tasks that promote reasoning and problem solving.

Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.

Use and connect mathematical representations.

Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.

Facilitate meaningful mathematical discourse.

Effective teaching of mathematics facilitates discourse among students to build share understanding of mathematical ideas by analyzing and comparing student approaches and arguments.

Pose purposeful questions.

Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.

Build procedural fluency from conceptual understanding.

Effective teaching of mathematics builds -fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.

Support productive struggle in learning mathematics.

Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.

Elicit and use evidence of student thinking.

Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

[1] National Council of Teachers of Mathematics (2014). *Principles to Actions: Ensuring Mathematical Success for All.* Reston, VA.

TABLE 2. Common multiplication and division situations [2]

	Unknown Product $3 \times 6 = ?$	Group Size Unknown ("How many in each group?" Division) $3 \times ? = 18$ and $18 \div 3 = ?$	Number of Groups Unknown ("How many groups?" Division) $? \times 6 = 18$ and $18 \div 6 = ?$
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example*. You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example*. You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example*. You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays,Area	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example*. What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example*. A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example*. A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs \$6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example*. A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs \$18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example*. A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs \$18 and a blue hat costs \$6. How many times as much does the red hat cost as the blue hat? *Measurement example*. A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	$a \times b = ?$	$a \times ? = p$ and $p \div a = ?$	$? \times b = p$ and $p \div b = ?$

[2] Mathematics Glossary, Common Core State Standards @ http://www.corestandards.org/