# Session One Overview

## Making Meaning for Operations: Session 1

### Agenda

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|---|---|---|
| Orientation | Whole group | 5 minutes |
| Comments about the Common Core | Whole group | 15 minutes |
| Chapter 1 Case Discussion (part 1) | Small groups<br>Whole group | 20 minutes<br>15 minutes |
| Math Activity: Modeling Story Problems | Small groups<br>Whole group | 20 minutes<br>15 minutes |
| Break | | 15 minutes |
| DVD for Session 1 | Whole group | 15 minutes |
| Chapter 1 Case Discussion (part 2) | Small groups<br>Whole group | 30 minutes<br>25 minutes |
| Homework and Exit Cards | Whole group | 5 minutes |

### Mathematical Themes

- Young children can solve problems by counting before they learn to add and subtract; through solving such problems, they begin to develop meaning for the operations.
- The same situation can be represented by an addition and a subtraction sentence.
- Different kinds of situations can be represented by the same subtraction expressions.

<u>Connections to the Common Core: Standards for Mathematical Practice</u>

MP2 Reason abstractly and quantitatively.

MP3 Construct viable arguments and critique the reasoning of others.

<u>Connections to the Common Core: Content Standards</u>

Kindergarten: Counting and Cardinality 2, 4 and 5
Operations and Algebraic Thinking 1 and 2
Grade 1: Operations and Algebraic Thinking 1, 3, 4, 5,6, and 8
Grade 2: Operations and Algebraic Thinking 1

*K.CC.2. Count forward beginning from a given number within the known sequence (instead of having to begin at 1).*

*K.CC.4. Understand the relationship between numbers and quantities; connect counting to cardinality.*

*K.CC.5. Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.*

*K.OA. 1. Represent addition and subtraction with objects, fingers, mental images, drawings, sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.*

*K.OA.2. Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.*

*1.OA.1. Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions; e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.*

*1.OA.3. Apply properties of operations as strategies to add and subtract.3 Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Associative property of addition.)*

*1.OA.4. Understand subtraction as an unknown-addend problem. For example, subtract 10 – 8 by finding the number that makes 10 when added to 8.*
*1.OA.5. Relate counting to addition and subtraction (e.g., by counting on 2 to add 2).*

*1.OA.6. Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 – 4 = 13 – 3 – 1 = 10 – 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 – 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13).*

*1.OA.8. Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations 8 + ? = 11, 5 = □ – 3, 6 + 6 = □.*

*2.OA.1. Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions; e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.*

Notes to the Facilitator regarding the Standards for Mathematical Practice

**MP2** Reason abstractly and quantitatively. *Mathematically proficient students at the elementary grades make sense of quantities and their relationships in problem situations. They can contextualize quantities and operations by using images or stories. They interpret symbols as having meaning, not just as directions to carry out a procedure. Even as they manipulate the symbols, they can pause as needed to access the meaning of the numbers, the units, and the operations that the symbols represent.*

When students connect story contents, number sentences, and other representations to make meaning for symbols and develop an understanding of mathematical structures, they are engaged in MP2. In the cases of chapter 1, students work with a variety of stories and representations to develop new insights into addition and subtraction. During the case discussion, highlight participants' comments about how the students in the cases call upon the action of the story context to create their number sentences or solution strategies and point out that these are examples of enacting MP2. The section in Maxine's Journal, Mathematics Activity: Modeling Story Problems, illustrates participants engaged in MP2. While Maxine does not specifically refer to the mathematical practices, you should make the connections explicit.

**MP3** Construct viable arguments and critique the reasoning of others. *Mathematically proficient students at the elementary grades construct mathematical arguments—that is, explain the reasoning underlying a strategy, solution, or conjecture—using concrete referents such as objects, drawings, diagrams, and actions…. Mathematically proficient students can listen to or read the arguments of others, decide whether they make sense, ask useful questions to clarify or improve the arguments, and build on those arguments. They can communicate their arguments, compare them to others, and reconsider their own arguments in response to the critiques of others.*

In case 6, the teacher, Machiko, asks the class, "How could we use Katie's idea to find more numbers that have a differences of 153?" This is an example, of how a teacher invites the class to engage in the second part of MP3: Critique the reasoning of others. Critique does not mean to criticize. Rather, it involves following another's reasoning. Case 6 is a good one to look at how the class together constructs a strategy for solving a set of problems by listening to each other and building on one another's thinking.

In the Math Activity: Modeling Story Problems, participants frequently disagree with each other's interpretations and solutions. As they work to understand each other's arguments and approaches, they are engaged in MP3. At times, critiquing the reasoning of others involves asking clarifying questions which may challenge one to rethink or revise the original idea.

**Facilitator Note on MP4: Highlighting a common misconception**

**MP4** Model with mathematics: *When given a problem in a contextual situation, mathematically proficient students at the elementary grades can identify the mathematical elements of a situation and create a mathematical model that shows those mathematical elements and relationships among them.*

Note that the word, *model*, has two different definitions that apply in mathematics education. Frequently *model* indicates a physical representation, such as the cube structures or counters used by some of the students in case 5 to work on the valentine sticker problem. These physical embodiments of mathematics are powerful tools to support the reasoning of participants and their students. However they are not examples of MP4.

A second definition of *model* applies to MP4: a description or representation of a situational context using mathematical concepts. In order to fit the meaning of MP4, the problem begins outside of mathematics; the mathematical aspects are abstracted from the context and expressed in mathematical language. The homework assignment in Session 3 includes an example of mathematical modeling. In this case, participants work with a contextual situation, 7 brownies to be shared among 4 people. As they reason about the situation, they express the mathematics in the situation with a division expression, $7 \div 4$, modeling the situation with numbers and symbols.

## MMO Session One Agenda Changes linked to Common Core

There are six modifications to Session One.

1. At the opening of the session, distribute the overview for the session. Let the group know you will share a document like this at each session. After giving them a few minutes to look this over, comment that some of the language on the overview might be unfamiliar. Explain that during the sessions, you will periodically point to the practice or content standards and link them to the work of the session.

2. Replace the discussion of "Sharing Student Work Samples" with this discussion labeled "Comments about the Common Core."

Comments about the Common Core (whole group) (15 minutes)

Let participants know that one component of the seminar will be to highlight connections to the Common Core Content and Standards for Mathematical Practice. Distribute the elementary elaborations of the Standards for Mathematical Practice (unless you have provided these to them as a pre-seminar reading). Explain that while all the practice standards are important and that they will experience all of them during the seminar, two practices will be particularly highlighted: MP2 and MP3. Give participants 10 minutes to read these two practices.

After the reading time, let participants know you are not expecting them to fully comprehend these now, but rather to begin the work of having them make sense. Throughout the seminar, they will reflect on their own work and the work of the teachers and students in the print and video cases through the lens of these standards.

3. Two reflections have been added to the Focus Questions to support small group discussion of the standards for mathematical practice. The Focus Questions sheet has also been edited to align line numbers in the casebook with the questions. See below for a copy of the new Focus Questions.

4. Replace "Classification of Word Problems" with a table from the Common Core: Table 1. Common Addition and Subtraction Strategies. See below for a copy of this table.

5. A new exit question has been added to encourage participants to reflect on the standards for mathematical practice and for facilitators and staff to gather information on what participants are noticing. Exit Questions for Session One:

- What mathematical ideas did this session highlight for you?
- What was the session like for you as a learner?
- What questions or comments about MP2 came up for you at the session?

6. The writing assignment for Session Two has been edited to include a reference to the Standards for Mathematical Practice. See below for a copy of the revised assignment sheet.

## SESSION 1

**Focus Questions: Chapter 1**

**Case discussion, part 1**

1. In Denise's case 3, Susan works on a problem, getting an answer of 6 one day and 7 the next (lines 235 to 240). What is the logic behind each answer? What is the mathematical issue involved in this problem?

2. In case 1, Dan asks, "When do children move from merely counting to actually understanding addition?" What does Dan mean by this? Do you see children who understand addition in these cases? Explain your examples. Do you see children who are "merely" count in these cases? Explain your examples.

Reflection: Locate one or two examples of students engaged in MP3 in case 3, Denise's case. Explain how your choice illustrates MP3.

**Case discussion, part 2**

3. In Jody's case, case 5, the children display a variety of ways of solving a single problem. Examine the methods of Latasha, Jessie, Maya, and Antoinette. What does each student's method show you about his or her mental image of the problem? What connections do you see between the action in the story problem and the ways each student represents it?

4. One of Machiko's students in case 6, Alex, says you can add or subtract to find numbers with a difference of 153. How can that be true if the term *difference* is meant to indicate the result of subtraction?

5. In case 4, Kina relates a story about the thinking of Zenobia. What connections do you see between the problem Zenobia is working on in lines 260 and the game that Kina refers to in lines 280 to 285? What connections do you see between this case and Table 1. Common Addition and Subtraction Situations?

Reflection: Locate one or two examples of students engaged in MP2 in case 5, Jody's case. Explain how your choice illustrates MP2.

6. The students in Machiko's case 6 use number lines to express their solutions. Discuss the methods of Donny (Figure 1.1), Reesa (Figure 1.2), Brad (Figure 1.3), Katie (Figure 1.4), and Alex (Figure 1.5). How are they the same and how are they different?

7. In the fourth-grade class presented in case 6, the students encounter negative numbers as they work to find pairs of numbers with a difference of 153. What ideas about addition, subtraction, and negative numbers do they call upon as they make sense of this?

## **TABLE 1**. Common addition and subtraction situations [1]

| | Result Unknown | Change Unknown | Start Unknown |
|---|---|---|---|
| **Add to** | Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now?<br>$2 + 3 = ?$ | Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two?<br>$2 + ? = 5$ | Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before?<br>$? + 3 = 5$ |
| **Take from** | Five apples were on the table. I ate two apples. How many apples are on the table now?<br>$5 - 2 = ?$ | Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat?<br>$5 - ? = 3$ | Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before?<br>$? - 2 = 3$ |

| | Total Unknown | Addend Unknown | Both Addends Unknown |
|---|---|---|---|
| **Put Together/ Take Apart** | Three red apples and two green apples are on the table. How many apples are on the table?<br>$3 + 2 = ?$ | Five apples are on the table. Three are red and the rest are green. How many apples are green?<br>$3 + ? = 5$, $5 - 3 = ?$ | Grandma has five flowers. How many can she put in her red vase and how many in her blue vase?<br>$5 = 0 + 5$, $5 = 5 + 0$<br>$5 = 1 + 4$, $5 = 4 + 1$<br>$5 = 2 + 3$, $5 = 3 + 2$ |

| | Difference Unknown | Bigger Unknown | Smaller Unknown |
|---|---|---|---|
| **Compare** | ("How many more?" version):<br>Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy?<br>("How many fewer?" version):<br>Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie?<br>$2 + ? = 5$, $5 - 2 = ?$ | (Version with "more"):<br>Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have?<br>(Version with "fewer"):<br>Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have?<br>$2 + 3 = ?$, $3 + 2 = ?$ | (Version with "more"):<br>Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have?<br>(Version with "fewer"):<br>Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have?<br>$5 - 3 = ?$, $? + 3 = 5$ |

1 Mathematics Glossary, Common Core State Standards @ http://www.corestandards.org/

## SESSION 1

## Second Homework

### Reading assignment: Casebook chapter 2

In the casebook, read chapter 2, "Making Meaning for Multiplication and Division," including the introductory text and cases 8–12. Consider the questions posed in the introduction to this chapter as you read the cases.

### Writing assignment: Examples of student thinking

It is likely that reading the cases and working on the mathematics in this seminar have made you curious about how your own students think about the mathematics they do. This assignment asks you to examine the thinking of your students as they work on the two types of problems shown in the DVD clip. Choose numbers that are appropriate for the grade level you teach.

1. I have 375 (or 55, or 15) candy bars. I sell 90 (or 30, or 8) of them. How many candy bars do I have left?
2. I am taking a trip to visit my sister. I drive 90 (or 30, or 8) miles and then stop to rest. The total distance to my sister's house is 375 (or 55, or 15) miles. How much farther do I have to go?

Ask your students to

- make a number line model for each problem.
- write a number sentence for each problem.
- solve the number sentences.
- explain how the two problems are similar and how they are different.

After the session, think about what happened. What did you expect? Were you surprised? What did you learn? As you listened to the class session, did you notice your students engaged in any of the Standards for Mathematical Practice?

Write up your question, how your students responded, and what you make of their responses. Include specific examples of student work or dialogue. Include comments on one of the standards for mathematical practice you noticed. Reporting in detail about the work of a few students is very helpful. In particular, it is useful to analyze the work of students whose work might be confusing.

At our next session, you will share this writing with colleagues in the seminar. Please bring three copies of your writing to share and to turn in.

*Note:* You will be asked to prepare similar assignments that involve investigating students' thinking in preparation for Session Four and Session Six. Check your classroom schedules and lesson plans to make time to complete these assignments.

# Mathematics Standards for Mathematical Practice
# Elementary Elaborations [2]

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

**1 Make sense of problems and persevere in solving them.**

Mathematically proficient students at the elementary grades explain to themselves the meaning of a problem, look for entry points to begin work on the problem, and plan and choose a solution pathway. For example, young students might use concrete objects or pictures to show the actions of a problem, such as counting out and joining two sets to solve an addition problem. If students are not at first making sense of a problem or seeing a way to begin, they ask questions that will help them get started. As they work, they continually ask themselves, "Does this make sense?" When they find that their solution pathway does not make sense, they look for another pathway that does. They may consider simpler forms of the original problem; for example, to solve a problem involving multidigit numbers, they might first consider similar problems that involve multiples of ten or one hundred. Once they have a solution, they look back at the problem to determine if the solution is reasonable and accurate. They often check their answers to problems using a different method or approach.

Mathematically proficient students consider different representations of the problem and different solution pathways, both their own and those of other students, in order to identify and analyze correspondences among approaches. They can explain correspondences among physical models, pictures or diagrams, equations, verbal descriptions, tables, and graphs.

**2 Reason abstractly and quantitatively.**

Mathematically proficient students at the elementary grades make sense of quantities and their relationships in problem situations. They can contextualize quantities and operations by using images or stories. They interpret symbols as having meaning, not just as directions to carry out a procedure. Even as they manipulate the symbols, they can pause as needed to access the meaning of the numbers, the units, and the operations that the symbols represent. Mathematically proficient students know and flexibly use different properties of operations, numbers, and geometric objects.

Mathematically proficient students can contextualize an abstract problem by placing it in a context they then use to make sense of the mathematical ideas. For example, when a student sees the expression 40 – 26, she might visualize this problem by thinking, if I have 26 marbles and Marie has 40, how many more do I need to have as many as Marie? Then, in that context, she thinks, 4 more will get me to a total of 30, and then 10 more will get me to 40, so the answer is 14. In this example, the student uses a context to think through a strategy for solving the problem, using the relationship between addition and subtraction and decomposing and recomposing the quantities. She then uses what she did in the context to identify the solution of the original abstract problem.

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[2] Illustrative Mathematics. (2014, February 12). Standards for Mathematical Practice: Commentary and Elaborations for K–5. Tucson, AZ.For discussion of the Elaborations and related topics,see the Tools for the Common Core blog: http://commoncoretools.me.

Mathematically proficient students can also make sense of a contextual problem and express the actions or events that are described in the problem using numbers and symbols. If they work with the symbols to solve the problem, they can then interpret their solution in terms of the context. For example, to find the area of the floor of a rectangular room that measures 10 m. by 12 m., a student might represent the problem as an equation, solve it mentally, and record the problem and solution as $10 \times 12 = 120$. He has *decontextualized* the problem. When he states at the end that the area of the room is 120 square meters, he has *contextualized* the answer in order to solve the original problem. Problems like this that begin with a context and are then represented with mathematical objects or symbols are also examples of modeling with mathematics (MP.4).

**3 Construct viable arguments and critique the reasoning of others.**

Mathematically proficient students at the elementary grades construct mathematical arguments—that is, explain the reasoning underlying a strategy, solution, or conjecture—using concrete referents such as objects, drawings, diagrams, and actions. For example, in order to demonstrate what happens to the sum when the same amount is added to one addend and subtracted from another, students in the early grades might represent a story about children moving between two classrooms: the number of children in each classroom is an addend; the total number of children in the two classrooms is the sum. When some students move from one classroom to the other, the number of students in each classroom changes by that amount—one addend decreases by some amount and the other addend increases by that same amount—but the total number of students does not change. An older student might use an area representation to show why the distributive property holds. Arguments may also rely on definitions, previously established results, properties, or structures. For example, a student might argue that two different shapes have equal area because it has already been demonstrated that both shapes are half of the same rectangle. Students might also use counterexamples to argue that a conjecture is not true—for example, a rhombus is an example that shows that not all quadrilaterals with 4 equal sides are squares; or, multiplying by 1 shows that a product of two whole numbers is not always greater than each factor.

Mathematically proficient students present their arguments in the form of representations, actions on those representations, and explanations in words (oral or written). In the elementary grades, arguments are often a combination of all three. Some of their arguments apply to individual problems, but others are about conjectures based on regularities they have noticed across multiple problems (see MP.8, Look for and express regularity in repeated reasoning). As they articulate and justify generalizations, students consider to which mathematical objects (numbers or shapes, for example) their generalizations apply. For example, young students may believe a generalization about the behavior of addition applies to positive whole numbers less than 100 because those are the numbers with which they are currently familiar. As they expand their understanding of the number system, they may reexamine their conjecture for numbers in the hundreds and thousands. In upper elementary grades, students return to their conjectures and arguments about whole numbers to determine whether they apply to fractions and decimals. For example, students might make an argument based on an area representation of multiplication to show that the distributive property applies to problems involving fractions.

Mathematically proficient students can listen to or read the arguments of others, decide whether they make sense, ask useful questions to clarify or improve the arguments, and build on those arguments. They can communicate their arguments, compare them to others, and reconsider their own arguments in response to the critiques of others.

**4 Model with mathematics.**

When given a problem in a contextual situation, mathematically proficient students at the elementary grades can identify the mathematical elements of a situation and create a mathematical model that shows those mathematical elements and relationships among them. The mathematical model might be represented in one or more of the following ways: numbers and symbols, geometric figures, pictures or physical objects used to abstract the mathematical elements of the situation, or a mathematical diagram such as a number line, a table, or a graph, or students might use more than one of these to help them interpret the situation.

For example, when students are first studying an operation such as addition, they might arrange counters to solve problems such as this one: there are seven animals in the yard, some are dogs and some are cats, how many of each could there be? They are using the counters to model the mathematical elements of the contextual problem—that they can split a set of 7 into a set of 3 and a set of 4. When they learn how to write their actions with the counters in an equation, $4 + 3 = 7$, they are modeling the situation with numbers and symbols. Similarly, when students encounter situations such as sharing a pan of cornbread among 6 people, they might first show how to divide the cornbread into 6 equal pieces using a picture of a rectangle. The rectangle divided into 6 equal pieces is a model of the essential mathematical elements of the situation. When the students learn to write the name of each piece in relation to the whole pan as 1/6, they are now modeling the situation with mathematical notation.

Mathematically proficient students are able to identify important quantities in a contextual situation and use mathematical models to show the relationships of those quantities, particularly in multistep problems or problems involving more than one variable. For example, if there is a Penny Jar that starts with 3 pennies in the jar, and 4 pennies are added each day, students might use a table to model the relationship between number of days and number of pennies in the jar. They can then use the model to determine how many pennies are in the jar after 10 days, which in turn helps them model the situation with the expression, $4 \times 10 + 3$.

Mathematically proficient students use their model to analyze the relationships and draw conclusions. They interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

As students model situations with mathematics, they are choosing tools appropriately (MP.5). As they decontextualize the situation and represent it mathematically, they are also reasoning abstractly (MP.2).

## 5 Use appropriate tools strategically.

Mathematically proficient students at the elementary grades consider the tools that are available when solving a mathematical problem, whether in a real-world or mathematical context. These tools might include physical objects (cubes, geometric shapes, place value manipulatives, fraction bars, etc.), drawings or diagrams (number lines, tally marks, tape diagrams), paper and pencil, rulers and other measuring tools, scissors, tracing paper, grid paper, arrays, virtual manipulatives or other available technologies. Proficient students are sufficiently familiar with tools appropriate for their grade and areas of content to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained from their use as well as their limitations.

Mathematically proficient students choose tools that are relevant and useful to the problem at hand. These include such tools as are mentioned above as well as mathematical tools such as estimation or a particular strategy or algorithm. For example, in order to solve 3/5 – ½, a student might recognize that knowledge of equivalents of ½ is an appropriate tool: since ½ is equivalent to 2 ½ fifths, the result is ½ of a fifth or 1/10.

This practice is also related to looking for structure (MP.7), which often results in building mathematical tools that can then be used to solve problems.

## 6 Attend to precision.

Mathematically proficient students at the elementary grades communicate precisely to others. They start by using everyday language to express their mathematical ideas, realizing that they need to select words with clarity and specificity rather than saying, for example, "it works" without explaining what "it" means. As they encounter the ambiguity of everyday terms, they come to appreciate, understand, and use mathematical vocabulary. Once young students become familiar with a mathematical idea or object, they are ready to learn more precise mathematical terms to describe it. In using representations, such as pictures, tables, graphs, or diagrams, they use appropriate labels to communicate the meaning of their representation.

When making mathematical arguments about a solution, strategy, or conjecture (see MP.3), mathematically proficient students learn to craft careful explanations that communicate their reasoning by referring specifically to each important mathematical element, describing the relationships among them, and connecting their words clearly to their representations.

Elementary students learn to use mathematical symbols correctly and can describe the meaning of the symbols they use. In particular, they understand that the equal sign denotes that two quantities have the same value, and can use it flexibly to express equivalences. For example, the equivalence of 8 and 5 + 3 can be written both as 5 + 3 = 8 and 8 = 5 + 3. Similarly, the equivalence of 6 + 2 and 5 + 3 is expressed as 6 + 2 = 5 + 3.

When measuring, mathematically proficient students use tools and strategies to minimize the introduction of error. From Kindergarten on, they count accurately, using strategies so that they include each object once and only once without losing track. Mathematically proficient students calculate accurately and efficiently and use clear and concise notation to record their work.

## 7 Look for and make use of structure.

Mathematically proficient students at the elementary grades use structures such as place value, the properties of operations, other generalizations about the behavior of the operations (for example, the less you subtract, the greater the difference), and attributes of shapes to solve problems. In many cases, they have identified and described these structures through repeated reasoning (MP.8). For example, when younger students recognize that adding 1 results in the next counting number, they are identifying the basic structure of whole numbers. When older students calculate 16 × 9, they might apply the structure of place value and the distributive property to find the product: 16 × 9 = (10 + 6) × 9 = (10 × 9) + (6 × 9). To determine the volume of a 3 × 4 × 5 rectangular prism, students might see the structure of the prism as five layers of 3 × 4 arrays of cubes.

## 8 Look for and express regularity in repeated reasoning.

Mathematically proficient students at the elementary grades look for regularities as they solve multiple related problems, then identify and describe these regularities. For example, students might notice a pattern in the change to the product when a factor is increased by 1: 5 × 7 = 35 and 5 × 8 = 40—the product changes by 5; 9 × 4 = 36 and 10 × 4 = 40—the product changes by 4. Students might then express this regularity by saying something like, "When you change one factor by 1, the product increases by the other factor." Younger students might notice that when tossing two-color counters to find combinations of a given number, they always get what they call "opposites"—when tossing 6 counters, they get 2 red, 4 yellow and 4 red, 2 yellow and when tossing 4 counters, they get 1 red, 3 yellow and 3 red, 1 yellow. Mathematically proficient students formulate conjectures about what they notice, for example, that when 1 is added to a factor, the product increases by the other factor; or that, whenever they toss counters, for each combination that comes up, its "opposite" can also come up. As students practice articulating their observations, they learn to communicate with greater precision (MP.6). As they explain why these generalizations must be true, they construct, critique, and compare arguments (MP.3).